Chapters 6.1 Drawing graphs in the plane

Informally, drawing of a graph G in the plane is assignment of distinct points to the vertices and curves to edges such that curves have as endpoints their vertices and curves intersect only at endpoints. A very formal definition is in the textbook.

A graph G is **planar** if it is possible to draw it in the plane (without crossings of edges - except their endpoints).

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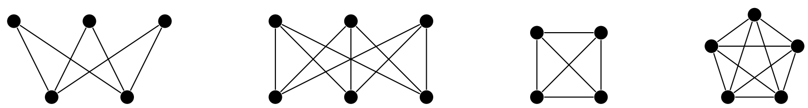
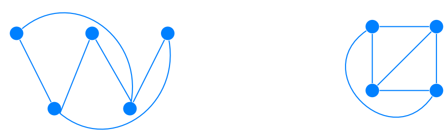
1: Are the following graphs $K_{2,3}$, $K_{3,3}$, K_4, and K_5 planar graphs?

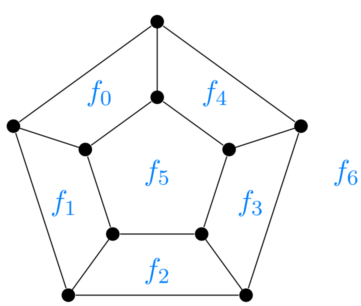


Solution: $K_{2,3}$ yes $K_{3,3}$ no K_4 yes K_5 no



A **face** (sometimes called **region**) in a plane graph G is a region of the plane that is obtained by removing the edges (and vertices) of G from the plane. (Imagine drawing G on a paper and cutting along the edges. The connected pieces of the paper after the cuttings is done are called faces.)

2: Mark individual faces in the following plane graph. How many faces does it have?



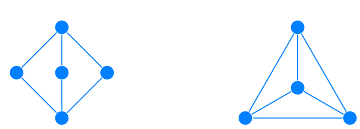
The unbounded piece is called **outer/exterior face/region**.

Drawings can be very wild, but there is always a *simple* one.

Theorem If G is a planar graph, then it has a drawing where all edges correspond to straight line segments.

3: Draw $K_{2,2}$ and K_4 using only straight lines.

Solution: One has to move vertices.



Theorem - Euler's Identity Let G be a *connected* plane graph with $v \geq 1$ vertices, e edges and f faces. Then

$$v + f = e + 2.$$

4: Verify that Euler's identity holds for $K_{2,3}$, K_4 and all trees. Notice that trees are planar graphs.

Solution: We will use drawings of K_4 and $K_{2,3}$ that we already have to count the faces. Denote the set of faces of G by $F(G)$.
$|V(K_4)| = 4$, $|F(K_4)| = 4$, $|E(K_4)| = 6$
$|V(K_{2,3})| = 5$, $|F(K_{2,3})| = 3$, $|E(K_{2,3})| = 6$
Recall that for every tree T, $|E(T) = |V(T)| - 1$ and notice that in any drawing $|F(T)| = 1$.

5: Prove Euler's Identity. Use induction and that every graph can be created from one vertex by adding leaves and edges.

Solution: If G has one vertex, zero edges and one face, the identity holds. If G has a cycle C, then removing one edge from the cycle decreases the number of edges by one and number of faces by one. If G has a vertex of degree one, then removing the vertex and its incident edge decreases the number of edges by one and number of vertices by one. Notice that both cases change both sides of the equation by one.

6: Let G be a plane graph with f faces and e edges, where $e \geq 2$. Show that $3f \leq 2e$.
Hint: Counting (edge side)-face incidences should do it.

Solution: Let x be the number of (edge side)-face incidencies. This way, every edge is incident with two faces (or one face twice if it is a bridge) and we get $2e = x$. On the other hand, the smallest face is a triangle, hence $3f \leq x$. This gives $3f \leq 2e$.

Theorem. If G is a planar graph of order at least 3, then

$$|E(G)| \leq 3|V(G)| - 6.$$

7: Prove Theorem.
Hint: Use Euler and get rid of f using previous question.

Solution: If $|E(G)| \leq 3$, the inequality holds. We use the Euler Identity $v+f = e+2$ and combine it with $3f \leq 2e$. That gives $3v + 3f = 3e + 6$ and $2e + 3v \geq 3e + 6$, which is the same as $e \leq 3v - 6$. What happens if G is not connected?

8: Show that K_5 is not a planar graph.
Hint: Use Euler's Formula

Solution: K_5 has 10 edges and 5 vertices. So it does not satisfy that $10 \leq 3 \cdot 5 - 6$.

9: Show that every planar graph has a vertex of degree at most 5.
Hint: Use Eulers formula and the handshaking lemma.

Solution: Suppose for contradiction that G has a minimum degree 6. Then $2|E(G)| = \sum_{v \in V(G)} \deg v \geq 6|V(G)|$, which contradicts that $|E(G)| \leq 3|V(G)| - 6$.

10: Show that if G is a bipartite planar graph with at least one two edges then

$$|E(G)| \leq 2|V(G)| - 4.$$

Solution: Let G be a bipartite plane graph with f faces, v vertices and e edges,

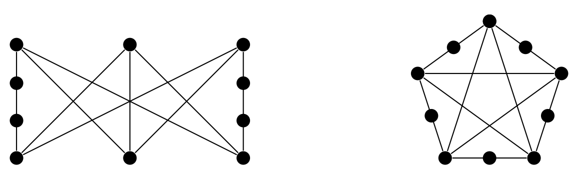
If G is bipartite and not trivial, then the smallest face has at least 4 edges. Hence $4f \leq 2e$.

We use the Euler Identity $v + f = e + 2$ and combine it with $4f \leq 2e$. That gives $2v + 2f = 2e + 4$ and $e + 2v \geq 2e + 4$, which is the same as $e \leq 2v - 4$.

11: Show that $K_{3,3}$ is not a planar graph.

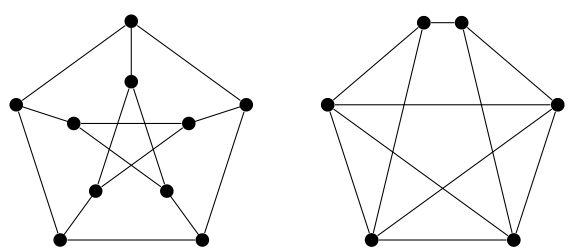
Solution: $K_{3,3}$ has 9 edges and 6 vertices. Hence it does not satisfy $e \leq 2v - 4$.

12: Are the following graphs planar? Why?



Solution: No. If they were planar, one could draw K_5 or $K_{3,3}$ as a planar graph.

13: Is Petersen's graph planar? And the one next to it?



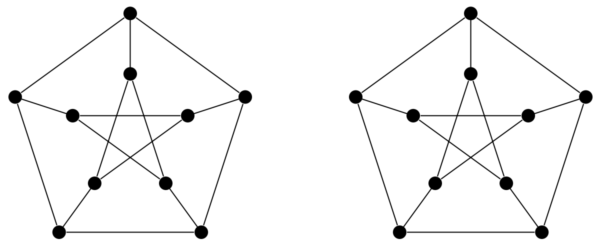
Solution: Petersen contains a subdivision of $K_{3,3}$. The other one looks basically like K_5. See the minor explanation below.

Theorem - Kuratowski A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

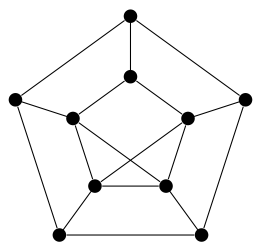
Let G be a graph. A graph H is a **minor** of G if H can be obtained from G by deleting vertices, deleting edges and contracting edges.

Theorem A graph G is planar iff it does not contain K_5 or $K_{3,3}$ as a minor.

14: Show that Petersen's graph has K_5 as a minor and also $K_{3,3}$ as a minor.



15: Are the following graphs planar?



16: Is it true that every bipartite planar graph has a vertex of degree three or less?

Solution:

A graph G is **maximal planar** if G is planar but addition of any edge makes G not planar.

17: Show that all faces of maximal planar graph are triangles.

Solution:

18: Let G be maximal planar graph of order 100 embedded in the plane. How many faces does it have?

Solution:

Crossing number of the graph is the minimum number of crossings of edges in a drawing of a graph in the plane. Note planar graphs have crossing number 0.

19: Show that crossing number of K_5 and $K_{3,3}$ is 1. That means, draw K_5 and $K_{3,3}$ in the plane such that there is just only in drawing of each of these two.

Solution:

20: Show that crossing number of K_6 is 3. That means, draw K_6 in the plane such that there are only three crossing. Argue that it cannot be done with 2 crossings.

Solution: Recall formula $e \leq 3v - 6$. For K_6, we would get

$$15 = \binom{6}{2} \leq 3 \cdot 6 - 6 = 12$$

It has 3 too many edges.

21: [Open problem] Prove that every planar graph of order n contains an independent set of size at least $n/4$. (Without using 4-color theorem. Best know is $\frac{3n}{13}$.)

22: [Open problem] Find a formula for a crossing number of K_n or $K_{n,n}$.

Planar graphs and Euler's formula

A graph is *planar* if we can draw it on the plane so that no two edges cross one another. (Note that we might have a drawing of the graph where the edges do cross; the point is there is *some* drawing where they do not cross.) Many simple graphs we have dealt with are planar, but we have also encountered many nonplanar graphs and in some sense most graphs are nonplanar. Examples of nonplanar graphs include K_5 and $K_{3,3}$ (we haven't shown they are nonplanar, but will soon).

Note that drawing a graph on the plane is equivalent to drawing the graph on a sphere indeed we can wrap the plane around the sphere, or more mathematically speaking we can project the plane to the sphere only missing a single point which we can easily touch up. Of course we can draw graphs on other surfaces and they have different behavior. For example, we can easily draw K_5 on a torus.

Given a planar graph there are the vertices and edges as before, but now we also have faces (i.e., imagine that we take the planar graph and cut along the vertices and edges, that cuts the plane into several pieces, and each piece is called a face; note that there is one huge face on the outside, called the unbounded face). A face is bounded by a set of edges and so the length of the face is how many edges are used to make up that face. Often times faces are grouped by how many edges make up the faces (note that if we are dealing with simple graphs that it takes at least three edges to make a face; a loop would only take one edge to make a face, and two edges joining the same pair of vertices only take two edges to make a face). If we let f_i denote the number of faces with i edges then in a simple graph we have the following:

$$f_3 + f_4 + f_5 + \cdots = |F| \quad \text{and} \quad 3f_3 + 4f_4 + 5f_5 + \cdots = 2|E|.$$

The first reflects the count of the number of faces and the second follows by noting that each edge will get used in exactly two faces.

Given a drawing of a planar graph we can construct its dual graph. This is done by putting a vertex in each face and then connecting two of these new vertices if the corresponding faces share an edge.

The most useful result for planar graphs is the following result.

Euler's Formula. *For a connected planar graph G we have $|V| - |E| + |F| = 2$.*

This has many proofs, here is a sketch of one. It is true for the graph K_1 ($|V| = |F| = 1$). Further, given it is true for a graph it is true if we add an edge between two existing vertices (increases $|E|$ and $|F|$ both by 1) or if we add a new vertex and an edge connecting that vertex to the graph (increases $|V|$ and $|E|$ both by 1). But any connected planar graph can be constructed by starting with K_1 and doing those steps, therefore the result is true.

Theorem 1. *For a simple planar graph $|E| \le 3|V| - 6$.*

Proof. From the above we have

$$2|E| = 3f_3 + 4f_4 + 5f_5 + \cdots \ge 3f_3 + 3f_4 + 3f_5 + \cdots = 3(f_3 + f_4 + f_5 + \cdots) = 3|F|.$$

Then by Euler's formula we have

$$2 = |V| - |E| + |F| \le |V| - |E| + \tfrac{2}{3}|E| \quad \text{so} \quad \tfrac{1}{3}|E| \le |V| - 2 \quad \text{or} \quad |E| \le 3|V| - 6. \qquad \square$$

From this we can immediately conclude that K_5 ($|V| = 5$ and $|E| = 10$) is not planar. A similar argument shows that $K_{3,3}$ is not planar. (I told you we would do this soon!)

More on planar graphs

Last time we discussed planar graphs (i.e., graphs which can be drawn in the plane (or sphere) so that no lines cross); these graphs have a new feature, faces. One of the most useful formulas that we have for working with connected planar graphs is Euler's formula: $|V| - |E| + F|F = 2$.

From this we can immediately arrive some simple consequences for (connected) planar graphs:

- $|E| \le 3|V| - 6$ and equality holds only if all faces are triangles.
- There is a vertex of degree ≤ 5.
- K_5 and $K_{3,3}$ are not planar.

Along with some other results (see homework for examples).

One question we have is when is a graph planar. From the above we see that if we have K_5 or $K_{3,3}$ as a subgraph we cannot be planar (i.e., every subgraph of a planar graph is planar). But more generally if we have a subgraph which has the same "structure" of K_5 or $K_{3,3}$ we cannot be planar. More precisely, we say that H is a minor of G if we can get from G to H by deleting and/or contracting edges. Note in particular if a graph has a $K_{3,3}$ or K_5 minor then it cannot be planar. It turns out this is sufficient.

Theorem 2 (Kuratowski (1930))**.** *A graph is planar if and only if it has no K_5 or $K_{3,3}$ minor.*

Given a graph the crossing number, $cr(G)$, is the minimum number of times that edges cross in some drawing of the graph on the plane. A graph is planar if and only if $cr(G) = 0$, on the other hand we have $cr(K_5) = 1$ and $cr(K_{3,3}) = 1$ and so these are non-planar. This idea was investigated by Turán who thought about this problem while in a World War II forced labor camp; he had to move bricks around and noticed that the most difficult part of the process was where two paths intersected and so considered the problem of minimizing the number of intersections of these paths. There are some easy bounds.

Theorem 3. $cr(G) \ge |E| - 3|V| + 6$.

Proof. Find a drawing with the least crossings and make it a planar graph by replacing each crossing with a vertex. This adds $cr(G)$ vertices and $2cr(G)$ edges. Since $|E| \le 3|V| - 6$ for any planar graph we can conclude for our original graph that $|E| + 2cr(G) \le 3(|V| + cr(G)) - 6$ and the result follows by rearranging. $\square$

By using more sophisticated techniques (i.e., ask a MATH 492 student at the end of the semester) it can be shown that $cr(G) \ge \frac{1}{64}\frac{|E|^3}{|V|^2}$. In general this leads to hard problems which are still open.

Conjecture 4. $cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$.

It's known "$\le$" holds and also is true if $\min\{m, n\} \le 6$, but that still leaves a lot to show!